A PROOF OF EUCLID'S 47th PROPOSITION
Using the Figure of □The Point Within a Circle□ and With the Kind Assistance of President James A. Garfield.

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American Sports Figure and inadvertent wit Yogi Berra is credited with saying □If you don't plan where you're going, you'll end up someplace else□. I frequently take exactly this sort of unplanned journey using the internet as my vehicle. More often than not my starting place is some Masonic concept or bit of symbolism which interests me, and on especially good journeys I actually end up discovering things which a more rigid, planned approach would have caused me to by-pass. The paper you are about to read is the product of just this sort of journey.

While performing internet based research on the symbolism of the □Point Within a Circle□, I happened upon a link to a website which offers various mathematical proofs of Euclid's 47th Proposition. One of these proofs immediately caught my eye, since it had been developed by Brother James A. Garfield, the twentieth President of the United States, and a Freemason. Bro. Garfields' elegant and quite famousi[i] proof involves the construction of a trapezoid which is divided into three separate right trianglesii[ii] (Figure 1). Two of these triangles are congruent and one is an Isoceles triangle. Garfield demonstrated through algebraic means that the area of the trapezoid is equal to the sum of the areas of the three right triangles and thereby proved that $c^2=a^2 + b^2$. Garfields' proof, which I happened upon when I also had the □Point Within a Circle□ symbol fresh in my mind, led me to consider whether the figure of the □Point Within a Circle□ might be used to construct a similar or even identical proof. As will be demonstrated, the figure of a □Point Within a Circle□ can not only be used to construct a proof of Euclid's 47th Proposition, it leads to the exact same method in doing so as that which was published by President Garfield.

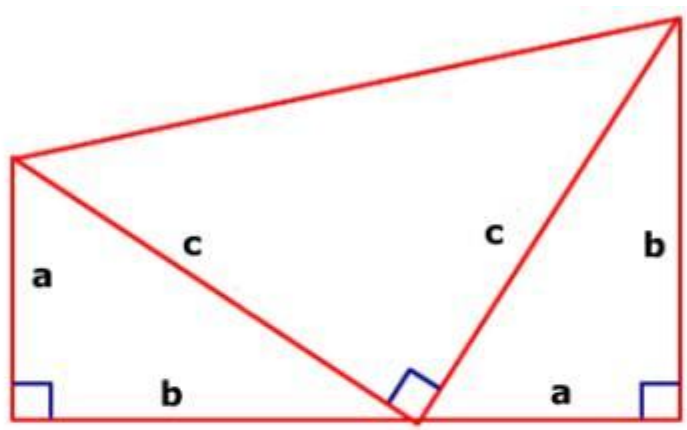


Figure 1 - The figure for the proof of Euclids' 47th Proposition used by President James A. Garfield. Note that the trapezoid is composed of three right triangles, one of which is an isosceles triangle and two of which are congruent. Garfield demonstrated that the combined areas of the three triangles equals the area of the trapezoid, establishing that $c^2 = a^2 + b^2$

In order to prepare the reader for this demonstration, I will present a brief and very basic discussion concerning

Euclid's 47th Proposition. Readers with a greater interest or who may be interested in a more complete discussion will find exhaustive reference material to this effect on the internet. I will follow with the construction of the diagram I used to demonstrate the proof in which the □Point Within a Circle□ is a critical element. Finally I will complete the algebraic portion of the proof using the exact same equations and mathematical methods used by Garfield. Please note that I do not claim this proof to be original in any way other than the fact that it was developed using the □Point Within a Circle□ as the basis for constructing the figure or diagram upon which Garfields' proof is based.

Euclid's 47th Proposition

During ones' journey through the rituals of Freemasonry, it is nearly impossible to escape exposure to Euclid's 47th Proposition and the Masonic symbol which depicts the proof of this amazing element of Geometry. Euclid's 47th Proposition of course presents what we commonly call the Pythagorean Theorem. The Pythagorean Theorem establishes that the square of the length of the hypotenuse in a right triangle will equal the square of the sums of the lengths of the other two sides. We state this mathematically as $c^2 = a^2 + b^2$ in which □c□ is the hypotenuse and □a□ and □b□ are the other two sides.

Although we identify the Pythagorean Theorem with the calculation of the length of the sides of a right triangle, its basis of proof is actually in the calculation of areas. The Pythagorean Theorem may be rephrased to state that the sum of the area of the squares enclosing two sides of a right triangle will equal the area of the square forming the side which is the hypotenuse of that triangle. One figure often used to establish the proof of this restated version of the Pythagorean Theorem is provided by Figure 2. Brilliant use is made in this figure of the first set of the Pythagorean Triples iii[iii] 3, 4, and 5.

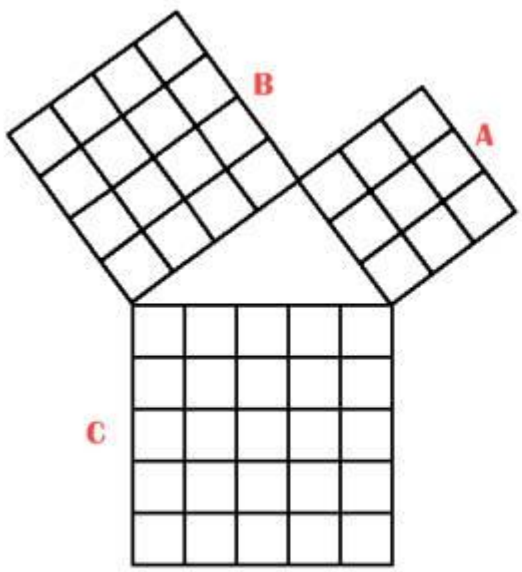


Figure 2 – Euclids' 47th Proposition, shown here with each of the squares enclosing the 3, 4, 5 triangle labeled as A, B, and C respectively. If you multiply the length of two sides of a square you are calculating that square's area. In our figure 3 X 3 = 9, 4 X 4 = 16, and 5 X 5 = 25. Accordingly 9 + 16 = 25.

Much is made of Euclid's 47th Proposition in Freemasonry, primarily in the third degree of the Craft. While the value of this Proposition to an Operative Mason is immediately apparent, its' meaning to the Speculative Mason is somewhat less so. The assumption of many Masons is that there is a great and abiding allusion contained within the Theorem, but this allusion is so heavily veiled or so subtle in meaning that it is incomprehensible. I personally continue to search for more light in Freemasonry through research in those areas which interest me. I might add that my efforts have never failed to deliver a greater understanding of the Craft.

I will begin my discussion of the proof of Euclid's 47th Proposition with the simple expedient of referring the reader to two seminal papers which describe the method for inscribing a right triangle in a circle in accordance with the Theorem of Thales. The first of these is contained in a paperiv[iv] presented during the 222nd Anniversary of Independent Royal Arch Lodge No. 2, F. & A.M. by Bro. Brent Morris. A second, also very detailed, paperv[v] describing the use of this method to construct a right triangle is given by Bro. William F. Bowe in The Builder Magazine. Both of these articles explain Euclid's Theorems: Theorem 12, contained in Book III of Euclid's Elementsvi[vi] in which it is stated that □An angle inscribed in a semi-circle is a right angle□. This Theorem is based upon an even older Theorem to the same effect developed by Greek Philosopher, Astronomer, and Mathematician Thales of Miletusvii[vii].

As stated, my demonstration makes exclusive use of the □Point Within a Circle□ to develop the proof figure introduced by President Garfield. Accordingly my initial step in this proof is to draw the □Point Within a Circle□ as a diagram, which I have done in Figure 3. I have added to this figure a straight line (□AB□) across the diameter of the circle and perpendicular to the two parallel lines (lines □CD□ and □EF□) at the points at which these lines are tangent to the circle.

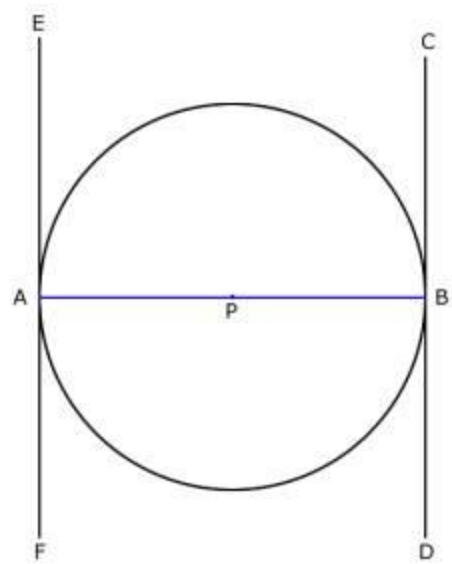


Figure 3 – The Masonic Symbol used to illustrate the "Point Within a Circle". A line across the diameter has been added which intersects the two parallel lines ("CD" and "EF") at the point at which they are tangent to the circle. Point "P" in the diagram is the center of the circle.

I next use Thales' Theorem to construct a right triangle (□ABV□) in the semicircle. I add line □PG□ which begins at the center point □P□ of the circle and which extends through the vertex of the right triangle (□point V□). Line □EH□ is then added which forms a perpendicular intersection at point □V□, establishing itself as a line tangent to the circle at this point.

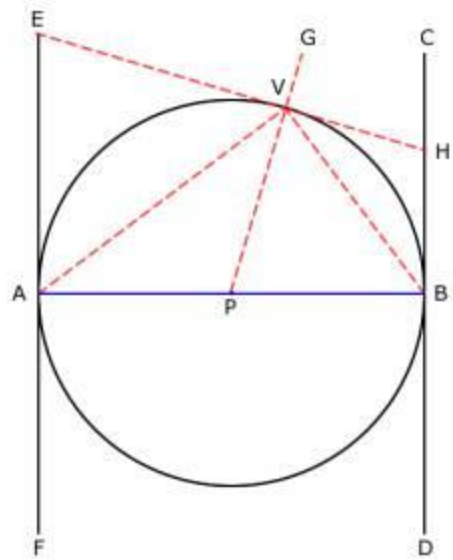


Figure 4 – The "Point Within a Circle with a right triangle inscribed using Thales' Theorem and with intermediate construction lines which provide line "EH" as a tangent to the triangles' vertex at point "V".

I now construct a line (□JN□)which forms a perpendicular intersection at the midpoint of line □EH□. Note that this line is parallel to line □PG□ and intersects the diameter line (□AB□) at point □N□. This is shown in Figure 5.

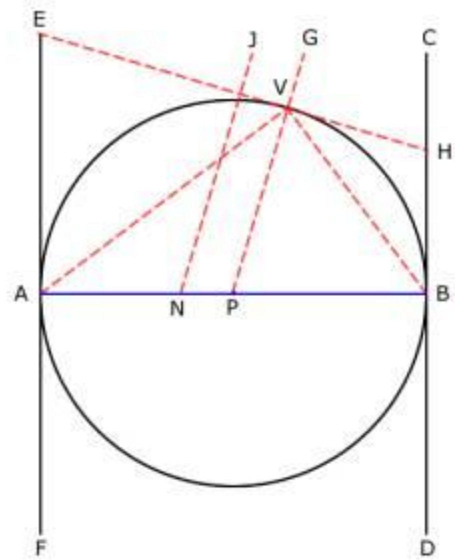


Figure 5 – A perpendicular bisector ("JN") has been constructed to the midpoint of line "EH" extending to and connecting to the diameter of the circle (line "AB") at point "N".

I next use a variation of Thales' Theorem to construct an Isosceles triangle by joining point □N□ with the points at which lines □CD□ and □EF□ are intersected by line □EH□. For those who are interested in further reading concerning this technique for creating a right triangle using a circle and tangent lines, I refer them to an articleviii[viii] published in Pietre-Stones Review of Freemasonry which goes into greater detail. Note that in constructing the isosceles triangle and the various construction lines I have simultaneously created in this figure a trapezoid (□ABHE□) composed of three right triangles, one of which is an isosceles triangle and two of which are congruent. In Figure 6 the trapezoid is outlined in blue for greater clarity.

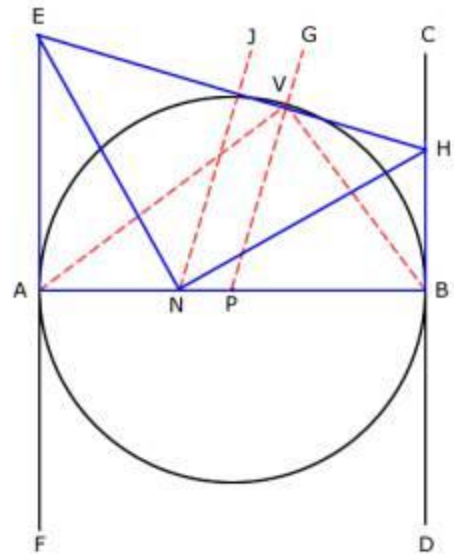
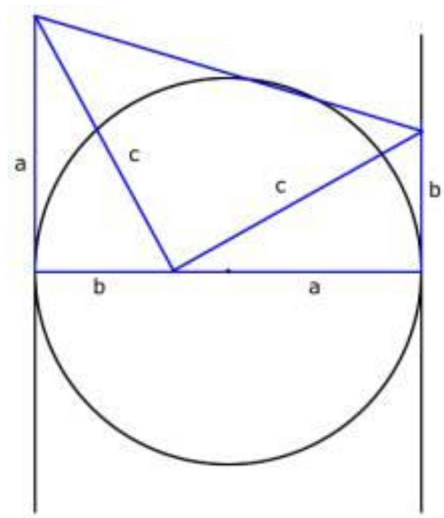


Figure 6 – An Isosceles triangle is constructed to complete the figure of proof. This simultaneously creates a trapezoid which is shown outlined in blue.

Figure 7 shows the trapezoid without the construction lines and extraneous labels. I have labeled the bases (□a□ and □b□) of the trapezoid, and the hypotenuses of the two right triangles (□c□). Note that the figure is nearly identical to that used by Bro. Garfield; although I have constructed the figure at hand with the slope of the trapezoid downhill. An uphill slope (making the figure truly identical to Garfields’) would have simply required construction of the Thales Triangle using a point left of center as the vertex. Incidentally Bro. Garfield was left-handedix[ix].



Using this proof figure and the associated labels I apply the exact same algebraic sequence for developing the proof as was applied by Bro. Garfield:

$$\frac{(a+b)}{2}(a+b) = \text{Area of Trapezoid}$$

$$\frac{ab}{2}+\frac{ab}{2}+\frac{c^2}{2} = \text{Area of Triangles Comprising Trapezoid}$$

$$\frac{2ab+c^2}{2} = \frac{(a+b)}{2}(a+b) \text{ Set area of Triangles Equal to Area of Trapezoid}$$

$$\frac{2ab+c^2}{2} = \frac{a^2+2ab+b^2}{2}$$

$$2ab+c^2 = a^2+2ab+b^2$$

$$c^2 = a^2+2ab-2ab+b^2$$

$$c^2 = a^2+b^2 \quad \blacksquare \text{ Resulting in the Pythagorean Theorem (End of Proof).}$$

Whether the President was ever aware that the Masonic symbol of the □Point Within a Circle□ could be used to prove Euclid's 47th Proposition in a manner so nearly identical to that which he demonstrated is of course unknown. It does however stir the imagination. President Garfield was, during his lifetime, a teacher of mathematics with a deep and abiding interest in Geometry (else there would likely be no Garfields’ Proof). As mentioned he was also a Freemason and would have been acquainted with the □Point Within a Circle□. I leave it to the reader to decide for himself whether Garfields’ proof was inspired by Masonic symbolism.